(From the Physical Institute of the German University in Prague)

Beziehungen zwischen klassischer Statistik und Quantenmechanik.

Relationships between classical statistics and quantum mechanics

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With the 4th edition. (Entered on January 19th, 1933)

The formal analogy between the differential equations for the position probability of a mechanical system according to classical statistics and quantum mechanics is elucidated, which can also be interpreted as equations for the movement of a visible swarm of particles of the same type, that defines a diffusion. The collisions of the molecules of the surrounding substance are recognized as physical causes for this diffusion in the classic case, and the uncertainty relationships in the quantum mechanical case. Force-free diffusion in the latter case is discussed and a simple derivation of the uncertainty relation based on this is given. The train of thought can be transferred to classical diffusion and one can derive an inequality for the scattering of positions and velocities that is closely analogous to Heisenberg's uncertainty relationship. The relationship found can also be transferred to a single particle and more generally to any mechanical system, where it states that the simultaneous measurement of position and associated speed is only possible up to a maximum accuracy because of the underlying Brownian motion. The relation of this result to the problem of the accuracy with which a physical quantity can be measured with a mechanical measuring instrument is discussed, whereby it is shown that, if the analysis is carried out analogously, there is an accuracy limit that cannot be exceeded here too. Finally, the question of why the classical diffusion equation is valid for a real density function with a real diffusion coefficient, while the Schrödinger equation is valid for a complex function with an imaginary diffusion coefficient, is examined from the point of view of wave mechanics and the problems of the observability of physical quantities and the reversibility or non-reversibility of the related to natural processes.

In the following, some relationships between classical statistics - the classical diffusion theory and the theory of Brownian motion - on the one hand and quantum mechanics on the other hand will be discussed, i.e. they arise for formal reasons and, although they may be known to some, in this context to my knowledge have not been addressed. In particular, it can be shown that Heisenberg's uncertainty relations can also be transferred to processes that are dominated by classical statistics, and that this opens up new perspectives on the frequently discussed question of the limits of what is possible to measure with a measuring instrument. Furthermore, an attempt is made to specify the physical meaning of the formal similarities and differences mentioned.

1. The classical diffusion theory is governed by the generalized diffusion equation[1]

$$\frac{\partial u}{\partial t} = D \cdot \nabla^2 u - \nabla \cdot (u\mathfrak{v}) \tag{1}$$

where u(x,y,z,t) is the concentration as a function of place and time, D is the diffusion coefficient (assumed to be constant) and $\mathfrak{v}$ is the velocity vector of the convection flow caused by external forces. The solution of this equation under given boundary conditions gives the concentration distribution at any future point in time if the distribution at the present point in time is known.

If one interprets the diffusion experiment as a collective experiment on a spatial entity with many particles of the same type, then $u\ dV$ is the relative frequency of those components of the entity that are found in the volume element dV at time t in this collective experiment, if u satisfies the normalization condition for all t.

$$\iiint u\ dV = 1 \tag{2}$$

If the space ensemble is exchanged for a virtual ensemble, the diffusion equation (1) turns into an equation for the "probability density" u of the position of an individual particle, which can be calculated as a function of time if it is known at time zero: Smoluchowski's differential equation for the Brownian motion of a single particle under the action of external forces.[2]

It can be shown that Smouluchowski's equation is a special case of another differential equation which can be derived under very general conditions for the Brownian motion of any mechanical system and is usually called the Fokker-Planck differential equation.3 It can be written according to Schrödinger in the form write,4 where F denotes a certain differential operator which, for the case where the system is a particle under the action of a force, reduces by (1) to $F = D\ \nabla^2 – \nabla \cdot \mathfrak{v}$.

$$\frac{\partial u}{\partial t} = Fu \tag{3}$$

As Schrödinger also pointed out,[1] the differential equation (3) is formally identical to the time-dependent Schrödinger differential equation of wave mechanics for the wave function ψ, which is usually written in the form where H denotes the Hamilton operator for the mechanical problem in question.

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi \tag{4}$$

According to the statistical view of wave mechanics, this equation is also a "probability equation" since it allows this variable to be calculated at any later point in time from the knowledge of ψ(q) at time zero, and the "probability amplitude" ψ with the probability density w for the stay of the system in a certain volume element of q-space is linked by the relation (ψ* conjugate of complex value to ψ),

$$w = \psi * \psi \tag{5}$$

provided that ψ satisfies the normalization condition.

$$\int \cdots \int \psi * \psi \; dV = 1 \tag{6}$$

Reversing the above train of thought, the quantity *w* given by (5) can also be understood here as the density of the phase points of a large number of similar systems in q-space that do not influence each other. Equation (4) then governs the variation of this density distribution and allows, from knowledge of the density function at time zero, to calculate the density at some later time.

In the particular special case in which the single system is a point mass of mass m which is under the action of forces derived from a potential U, equation (4) reads

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \tag{7}$$

The discussion of this equation teaches, as Ehrenfest was the first to show,[2] that the center of mass of a swarm of particles obeying the above conditions moves in ordinary, three-dimensional space as required by classical mechanics when the given forces act on the particles, but that in addition the particle swarm around this center of gravity spreads out by a kind of diffusion. In analogy to the movement of a particle swarm according to the classic diffusion equation (1), we have here a convection flow with superimposed diffusion.

Since we are only interested in the latter phenomenon, we want to set the external force field equal to zero in the following. Equations (1) and (7) then become formally identical viz

$$\frac{\partial u}{\partial t} = D \cdot \nabla^2 u \tag{8}$$

$$\frac{\partial \psi}{\partial t} = \varepsilon \cdot \nabla^2 u \tag{9}$$

Where the following substitution is used:

$$\varepsilon = \frac{i\hbar}{2m} \tag{10}$$

Under the same boundary and initial conditions, the solutions of (8) and (9) are therefore completely identical. An essential difference is the fact that in the quantum-mechanical case it is not the (generally complex) function ψ that occurs, but according to (5) its norm as a density function, and that the "diffusion coefficient" ε according to (10) is purely imaginary. We shall come back to the physical significance of this circumstance below.

2. The deeper reason for the analogy between the movement of a particle swarm according to the classical diffusion theory and according to quantum mechanics, which emerges in the comparison of §1, is that in both cases the speed of the individual particles of the swarm differs from one another and obeys a statistical law.

In the first case this is due to the fact that the particles are subjected to irregular collisions by the molecules of the surrounding substance, as a result of which the momentum of the particles varies continuously in magnitude and direction, so that there is no relation between the changes in momentum of different particles. This is noticeable when looking at a single particle in its irregular Brownian motion, when looking at a swarm of particles in the fact that, given the initial positions of the particles and the "macroscopically" measured initial velocity of zero, the particles actually have microscopic velocities that are irregularly distributed over the swarm and are themselves therefore the initial distribution changes over time in the manner characteristic of diffusion.

In the quantum mechanical case, the assumption of a certain initial density distribution of the particle swarm automatically means that the condition of vanishing initial velocities of all particles cannot be strictly fulfilled. Because according to Heisenberg's uncertainty principle, which fundamentally dominates quantum theory, a completely precise statement about the initial positions would be possible. Since certain information about the initial positions has already been conveyed by specifying the initial density distribution, a corresponding blurring of the initial velocity, i. H. a certain statistical distribution of the initial velocities of the swarm particles can be allowed. However, this necessarily means that after a certain time a change in the initial density distribution must have occurred as a "diffusion" of the swarm.

The fact that the inaccuracy of the information about the positions of the particles of the diffusing swarm together with the inaccuracy of the information about the velocities (or momenta) really satisfies Heisenberg's uncertainty principle was shown by Heisenberg, Kennard and others.[1,2] A brief derivation for the one-dimensional case may be given here, which, without using

transformation theory, makes use of equation (9) and the complex conjugate to it,[3] which reads in the one-dimensional case:

$$\left.\begin{aligned} \frac{\partial \psi}{\partial t} &= \varepsilon \frac{\partial^2 \psi}{\partial x^2} \\ \frac{\partial \psi^*}{\partial t} &= -\varepsilon \frac{\partial^2 \psi^*}{\partial x^2} \end{aligned}\right\} \tag{11}$$

Let x_0 be the initial position of a particle in the swarm, v its initial velocity and x its position after time t, then it holds

$$x = x_0 + vt \tag{12}$$

At time zero, the center of gravity falls in the origin of coordinates and its initial velocity is equal to zero, i.e. $\langle x_0 \rangle = 0$ and $\langle v \rangle = 0$. Then, according to (12), it is of course also $\langle x \rangle = 0$ for all t. By forming the square mean of (12) one obtains

$$\langle x^2 \rangle = \langle x_0^2 \rangle + 2\langle x_0 v \rangle + \langle v^2 \rangle t^2 \tag{13}$$

By definition

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \; dx \tag{14}$$

Using equation (11) and assuming that ψ vanishes at infinity in a sufficiently high order, one obtains from (14) after a simple calculation.

$$\frac{d\langle x^2 \rangle}{dt} = 2\varepsilon \int_{-\infty}^{+\infty} x \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right) \; dx \tag{15}$$

$$\frac{d^2\langle x^2 \rangle}{dt^2} = -8\varepsilon^2 \int_{-\infty}^{+\infty} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x} \right) \; dx \tag{16}$$

$$\frac{d^3\langle x^2 \rangle}{dt^3} = 0 \tag{17}$$

From (17) it follows that $\langle x^2 \rangle$ must be a quadratic function of time, in agreement with (13); for $\langle v^2 \rangle$ it follows as a coefficient of t^2 in (13) from (16)

$$\langle v^2 \rangle = \frac{1}{2} \frac{d^2\langle x^2 \rangle}{dt^2} = -4\varepsilon^2 \int_{-\infty}^{+\infty} \left| \frac{\partial \psi}{\partial x} \right|^2 \; dx \tag{18}$$

According to Heisenberg, the self-evident inequality (19) follows

$$\left| \frac{x}{2\langle x^2 \rangle} \psi + \frac{\partial \psi}{\partial x} \right|^2 \geq 0 \tag{19}$$

Using (6) and (14)

$$\int_{-\infty}^{+\infty} \left|\frac{\partial\psi}{\partial x}\right|^2 \, dx \geq \frac{1}{4\langle x^2\rangle}$$

And from here to (17)

$$\langle x^2\rangle\langle v^2\rangle \geq -\varepsilon^2 \tag{20}$$

Guides through to the relationships

$$\Delta x = \sqrt{\langle x^2\rangle} \tag{21a}$$

$$\Delta p = m\sqrt{\langle v^2\rangle} \tag{21b}$$

Considering the fuzziness of the determination of position and momentum in the particle swarm, then the Heisenberg relationship follows from (20) using (10) for ε

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{22}$$

The equals sign applies there if and only if the inequality (19) turns into an equation. The integration of the same gives for ψ

$$\psi = \text{const. } e^{-\frac{x^2}{4(\Delta x)^2}} \tag{23}$$

which is the probability density *w* of the particle swarm according to (5) with consideration of (6) the Gaussian distribution

$$w = \frac{1}{\sqrt{2\pi}\Delta x} \, e^{-\frac{x^2}{2(\Delta x)^2}} \tag{24}$$

If *ψ* has the special form (23) for $t = 0$, then it follows from (15) $\mathrm{d}\langle x^2\rangle/\mathrm{d}t=0$, and hence the coefficient of t vanishes in (13). If there is a corresponding initial distribution of the layers in the particle swarm under consideration, then $\langle x_0\, v\rangle=0$ is present in it at the same time, the scattering of the layers and that of the initial velocities of the individual particles are statistically independent of each other. Conversely, however, the existence of the density distribution (24) at zero time does not automatically mean that the positions and velocities are statistically dependent and therefore the linear term in (13) does not disappear either.

3. According to what was said at the beginning of §2, it makes sense to transfer the above consideration, which is based on Heisenberg's uncertainty principle in the quantum mechanical case, to the case of classical diffusion. Here, too, we limit ourselves to the one-dimensional case with vanishing convection flow, *i.e.* we start from equation (8), which reads in one dimension

$$\frac{\partial u}{\partial t} = D \cdot \frac{\partial^2 u}{\partial x^2} \tag{25}$$

where u according to (2) satisfies the normalization condition

$$\int_{-\infty}^{+\infty} u \, dx = 1 \tag{26}$$

We define the fuzziness of the particle swarm by the size $\langle x^2 \rangle$ according to

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 u \, dx \tag{27}$$

For $t = 0$, the center of gravity of the swarm should again be at the origin of the coordinates, i.e. $\langle x_0 \rangle = 0$ applies.

We first try to derive the analogue to equation (13), which expresses how the initial average squared displacement $\langle x_0^2 \rangle$ in the diffusing swarm of particles increases over time. Using equation (25) and assuming that u vanishes at infinity in a sufficiently high order, we first find:

$$\frac{d\langle x \rangle}{dt} = \frac{d}{dt} \int_{-\infty}^{+\infty} xu \, dx = \int_{-\infty}^{+\infty} x \frac{\partial u}{\partial t} \, dx = D \int_{-\infty}^{+\infty} x \frac{\partial^2 u}{\partial x^2} \, dx = 0$$

The center of gravity of the swarm therefore remains at rest, as is immediately evident from the absence of a convection flow that $\langle x \rangle = 0$ at all times.

From (27) it also follows in a similar way.

$$\frac{d\langle x^2 \rangle}{dt} = \frac{d}{dt} \int_{-\infty}^{+\infty} x^2 u \, dx = D \int_{-\infty}^{+\infty} x^2 \frac{\partial^2 u}{\partial x^2} \, dx = 2D \tag{28}$$

$\langle x^2 \rangle$ is therefore a linear function of the following form.

$$\langle x^2 \rangle = \langle x_0^2 \rangle + 2Dt \tag{29}$$

The comparison of (29) with (13) shows that in both cases, at least after a sufficiently long time, the blurring of the location increases indefinitely, so that a diffusion of the swarm takes place. However, while here the increase in $\langle x^2 \rangle$ occurs independently of $\langle x_0^2 \rangle$ and linearly with time, there the increase in time is *quadratic* and, because of the validity of (20), depends on $\langle x_0^2 \rangle$ itself (it occurs especially infinitely rapidly, if $\langle x_0^2 \rangle = 0$, since then $\langle v^2 \rangle$ becomes infinitely large); Finally, if the term linear in t in (13) does not vanish, *i.e.* the scattering of the positions and the velocities of the particles at zero time were not statistically independent of one another, the swarm can first contract to a minimum and only then can spread take place.

The formal reasons for the differences mentioned have already been mentioned at the end of §1. Physically, the differences can be explained by the fact that in the case of classical diffusion there is no “initial speed” of the particles and therefore no equation of the form (23), that rather

the speed of the particles is determined at any moment by the collisions of the molecules, as already mentioned was mentioned.[1]

If one assumes the statistical independence of the propagation process and the initial distribution in the classical case, one can write down equation (29) directly, since it expresses that the "square error" of *x*, which can be attributed to the two causes: initial scattering and diffusion, results from the two mentioned components together additively, whereby the second term represents the well-known Einstein's law for the mean square error of Brownian motion.

In order to find the analogue to the uncertainty relation (20) or (22), we first have to define a speed in a suitable way for the case of classical diffusion. From what has been said above, it is clear that this role can in any case not be played by the microscopic velocity caused by the collisions of the molecules with the particles. Likewise, as we have seen, the macroscopic speed of the particle swarm considered as a whole, more precisely the speed of its center of gravity, is zero. However, a size that is suitable for our purposes results from the consideration of the "diffusion current", *i.e.* the amount of diffusing substance passing in the diffusion field through a fixed area 1 in unit time.

The vector Q of the diffusion current is known1 as a position function in the diffusion field, which is linked to the scalar *u* by the following expression.

$$Q = -D\nabla u \tag{31}$$

We find the associated velocity vector $\mathfrak{v}$, since *u* is nothing else than the matter density of the diffusing substance, according to

$$\mathfrak{v} = \frac{1}{u}Q = -\frac{D}{u}\nabla u \tag{32}$$

Which in the one-dimensional case becomes

$$v = -D\frac{1}{u}\frac{\partial u}{\partial x} \tag{33}$$

Next we build the mean value of *v* of the particle swarm at any timepoint, then by definition using (25) we get

$$\langle v\rangle = \int_{-\infty}^{+\infty} vu\ dx = -D\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}\ dx = 0$$

As it must be, since ⟨v⟩ is nothing other than the macroscopic velocity of the center of gravity.

For the mean of $\langle v^2\rangle$ we find

$$\langle v^2 \rangle = \int_{-\infty}^{+\infty} v^2 u \, dx = D^2 \int_{-\infty}^{+\infty} \frac{1}{u} \left(\frac{\partial u}{\partial x} \right)^2 \, dx \tag{34}$$

For the product $\langle v^2 \rangle \cdot \langle x^2 \rangle$ one can now set up an inequality in direct transfer of the train of thought from §2 by again starting from the self-evident inequality

$$\left(\frac{1}{u} \frac{\partial u}{\partial x} + \frac{x}{\langle x^2 \rangle} \right)^2 \geq 0 \tag{35}$$

from which follows by solution

$$\frac{1}{u} \left(\frac{\partial u}{\partial x} \right)^2 \geq -\frac{2x}{\langle x^2 \rangle} \frac{\partial u}{\partial x} - \frac{x^2 u}{\langle x^2 \rangle^2}$$

By integration it follows from this using (26) and (27) after a simple calculation

$$\int_{-\infty}^{+\infty} \frac{1}{u} \left(\frac{\partial u}{\partial x} \right)^2 \, dx \geq \frac{1}{\langle x^2 \rangle}$$

and from this according to (34) finally

$$\langle x^2 \rangle \cdot \langle v^2 \rangle \geq D^2 \tag{36}$$

As can be seen, inequality (36) has exactly the same form as inequality (20), which becomes (36) if the absolute value of ε is replaced by *D* again.

By introducing the notations Δx and Δv in analogy to (21), we write our uncertainty relation in a simpler form

$$\Delta x \cdot \Delta v \geq D \tag{37}$$

The previous therefore states that in a classically diffusing swarm of particles, the positions and velocities of the particles cannot be determined arbitrarily precisely at any given moment, and that the product of the two uncertainties must always be greater than the diffusion coefficient *D*.

The lower limit is reached, *i. e*. the inequality turns into an equation if and only if the equals sign in (35) holds. The solution of the differential equation obtained in this way immediately yields, taking into account (26),

$$u = \frac{1}{\sqrt{2\pi} \cdot \Delta x} e^{-\frac{x^2}{2(\Delta x)^2}} \tag{38}$$

so again, as in the quantum mechanical case, the Gaussian distribution is in formal agreement with (24).

However, while the existence of the distribution (38) here necessarily results in the equation ΔxΔv=0, there the existence of the distribution (24) is a necessary but not a sufficient

condition for the product ΔxΔv to assume its smallest value. Furthermore, while here in a swarm of particles left to itself, which at time zero satisfies the minimum condition, this condition remains satisfied throughout the course of diffusion [because the distribution (38) is self-maintaining], in the quantum mechanical case the minimum condition is only in a single moment, e.g. at zero time, and not thereafter [because the shape of the distribution (23) is not maintained during the movement of the particles]. Finally, it should be emphasized that, in the classical case, a particle swarm corresponding to the minimum condition can always be imagined to have arisen by diffusion from one that was completely concentrated at a certain point in time in the origin of the coordinates. To see this, one only needs to insert (29) into (38) and introduce the abbreviation ⟨x02⟩ = 2Dt; you then get the formula

$$u = \frac{1}{2\sqrt{\pi D(t+t_0)}} e^{-\frac{x^2}{4D(t+t_0)}} \tag{39}$$

From which it follows that in fact u vanishes for $t = -t_0$ in the whole space with the exception of $x = 0$. In the quantum-mechanical case, as we have already remarked above, this reduction is not possible.

4. In the two previous paragraphs we discussed the application of the uncertainty relations in the quantum mechanical and in the classical case to a space ensemble of particles of the same kind. However, as is well known, the fundamental importance of the uncertainty principle in the quantum mechanical case is revealed when it is applied to a single system. It teaches that the simultaneous measurement of the position and the momentum of a force-free particle can at most be carried out with the accuracy $\hbar/2$ corresponding to formula (22), since the measuring process itself interferes with the measurement of one of the two quantities that the product of the inaccuracies of the two variables cannot fall below the value mentioned. In general, transferred to any mechanical system, the theorem can be expressed in such a way that the simultaneous measurement of a coordinate q and the momentum conjugated to it canonically is only possible with an inaccuracy of the order of magnitude h.

We can now easily transfer the relation (37) obtained in §3 to the problem of the simultaneous measurement of the position and speed of a particle that is under the effect of irregular collisions, *i.e.*, carrying out a Brownian movement, for the classic case. Our relationship teaches that the product of the inaccuracy of the position measurement and that of the simultaneously occurring velocity measurement cannot fall below the value D, where "velocity"

is understood to mean the macroscopic velocity of the particle, *i.e.*, the quantity $\delta x/\delta t$ (except that δt is large compared to the time between two consecutive molecular collisions against the particle). One sees that here too, as in the case of quantum mechanics, there is an impossibility of simultaneous precise measurement of position and velocity, which, however, is not determined by the process of measurement itself and regulated by a universal constant, as is the case there, but by the effect of the environment on the observed system is caused and is therefore of course not of a universal nature (*e.g.* by lowering the temperature, which is a factor in D, can be made arbitrarily small).

The fact that the formula (37) actually applies to measurements on a single particle can be made clear by the following consideration: We consider a force-free particle which is located at the origin of coordinates at time zero and has the macroscopic velocity zero . If we measure its position x after a short time t, the mean value $\langle x^2 \rangle$ of Einstein's formula suffices

$$\langle x^2 \rangle = 2Dt \tag{40}$$

$$\frac{d}{dt}\left(\frac{\langle x^2 \rangle}{2}\right) = D$$

If we interchange the differentiation according to time with the averaging, we get further

$$\left\langle \frac{d}{dt}\frac{x^2}{2} \right\rangle = \left\langle x\frac{dx}{dt} \right\rangle = D \tag{41}$$

x is now obviously the inaccuracy of the position of the particle caused by the Brownian movement (which we had brought to the position $x = 0$ at time zero), likewise dx/dt is the inaccuracy of the velocity caused by the same causes (which we made time zero equal zero). The product $\langle x\, dx/dt \rangle$ thus represents the inaccuracy product to be expected on average over many measurements. $\Delta x\, \Delta v$, which is equal to D according to equation (41). The fact that we have obtained exactly the minimum value here instead of equation (37) is due to the fact that the averaging refers to repeated measurements on a particle which we know, based on the assumption, that it always had the same initial position and initial velocity at time zero. It is immediately obvious that without knowledge of this circumstance, the inaccuracy can only increase, *i.e.* the product $\Delta x\, \Delta v$ can in fact be greater than D, as required by relation (37).Our relation states that when the accuracy of measuring the location of a Brownian particle increases, the accuracy of measuring the velocity decreases at the same time, and vice versa. Physically, one can understand the meaning of this statement by considering the following **Figures 1** to **4** in mind, of which Fig Function $x(t)$ observed with a

certain magnification, and **Figure 2** represents the function $v(t) = x'(t)$ obtained therefrom. **Figure 3** reproduces the beginning of **Figure 1**, observed with a higher magnification, and **Figure 4** again the velocity curve obtained therefrom. One sees immediately how the increase in the accuracy of position determination by increasing the magnification necessarily increases the uncertainty in the simultaneous determination of velocity. Our relations thus reproduce in exact form the fact, well known to anyone familiar with Brownian motion, that the trajectory of a Brownian particle exhibits more and more discontinuities the higher the magnification it is observed.

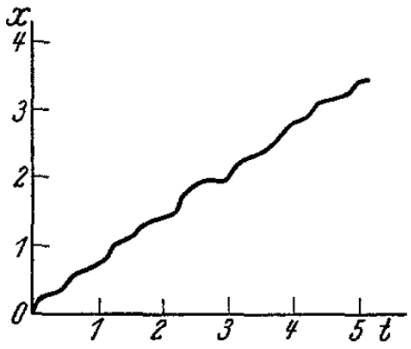

Fig. 1. Registrierkurve der Lage x eines Brownschen Teilchens als Funktion der Zeit t (schematisch).

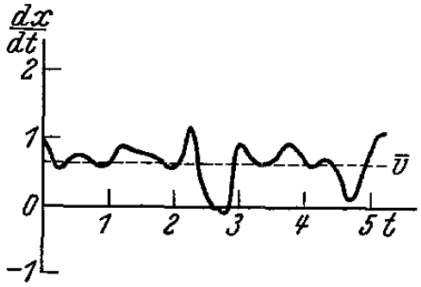

Fig. 2. Aus der Registrierkurve 1 ermittelte Geschwindigkeit v des Teilchens als Funktion der Zeit (- - - - mittlere Geschwindigkeit $\bar{v}$).

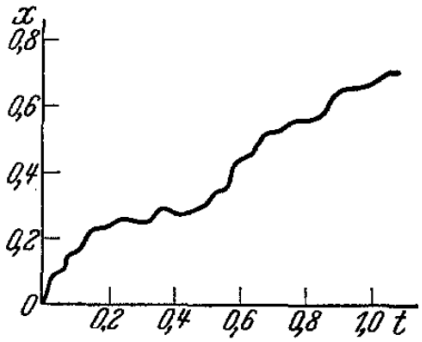

Fig. 3. Anfang der Registrierkurve 1, aufgenommen mit der 5fachen Genauigkeit für x (schematisch).

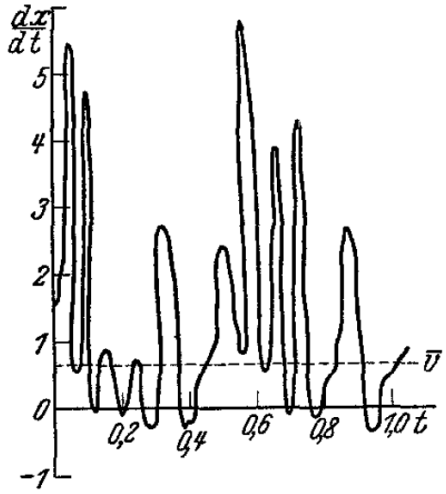

Fig. 4. Aus der Registrierkurve 3 ermittelte Geschwindigkeit v (- - - - mittlere Geschwindigkeit $\bar{v}$).

Just as in the quantum-mechanical case, we can transfer the uncertainty principle (37) to general mechanical systems of any kind that are in statistical-mechanical interaction with a

surrounding "temperature bath". In any case, each degree of freedom of movement corresponds to a Brownian movement of the associated coordinate, which we want to call x again. The corresponding Fokker-Planck equation (3) then takes the place of the differential equation (25) or (8).

It is plausible that in this general case, too, an uncertainty relation of the shape will apply

$$\Delta x \cdot \Delta v \approx D \tag{42}$$

where *v* is the velocity associated with coordinate *x* and *D* is the coefficient of the term $\partial^2 u'/\partial x2$ on the right side of equation (3) and is the characteristic constant of this general Brownian motion. Relation (42) states that the simultaneous measurement of the coordinate *x* and the associated velocity v is only possible with an uncertainty of the order of *D*.

5. Since in general every physical quantity, even if it is not of a mechanical nature, is measured by mechanical measuring instruments, *e.g.*, a current through a galvanometer formed from mechanical components. We can also extend the validity of formula (42) to any non-mechanical quantities. We assume that the "deflection" *x* of the measuring instrument used is proportional to the quantity J to be measured (*e.g.*, the deflection of a galvanometer is proportional to the current). If this is not the case from the outset, the desired condition can always be achieved within narrow limits by using a compensation method. The rate of change of *J* is *J'*. It then applies

$$J = ax, \qquad \dot{J} = a \cdot \dot{x} = av$$
$$\Delta J = a \cdot \Delta x, \quad \Delta \dot{J} = a \Delta v$$

And therefore using (42)

$$\Delta J \cdot \Delta \dot{J} \approx a^2 \cdot D \tag{43}$$

Relation (43) teaches that the accuracy of measurement can be increased at will by appropriate choice of the measuring instrument, namely by reducing a, but with a given measuring instrument by merely increasing the accuracy of the pointer reading, the accuracy of the simultaneous measurement of the quantity *J* and its rate of change cannot increase beyond a certain level because of the Brownian movement of the measuring instrument. Thus, in a moving-coil galvanometer with given mechanical properties, *a* can be reduced by increasing the magnetic field and therefore the accuracy of the current measurement can be increased at will, at least in principle; However, the product $\Delta J \cdot \Delta J'$ cannot be reduced simply by increasing the reading accuracy, for example by increasing the deflections using microscopic pointer reading [Ising][1] or by using the thermal relay [Moll][2] or the photoelectric relay [Bergmann][3].

The problem of the limits of the measuring accuracy of instruments, especially of galvanometers, due to the Brownian motion has been discussed repeatedly by various authors in recent times,[4] and there has been a detailed discussion as to which measures can be used to measure the quantity to be examined with an instrument as accurately as possible given design can cause. I think there is one important point that has always been overlooked in these discussions. The task of the experimenter is to register the variable J to be measured as a function of time, *i.e.*, to determine the function $J(t)$ with the greatest possible accuracy. Restricting oneself to a short time interval, this requirement is equivalent to the task of determining a variable J and its rate of change J' as precisely as possible in a given moment. Relation (43) teaches that with a given measuring instrument, this is only possible with a certain inaccuracy, regardless of any measures taken to increase the accuracy of the reading.

The method proposed by some authors, of increasing the measurement accuracy for J in spite of the Brownian motion, by taking many readings and taking the mean from them, which is then said to be more accurate than the individual measurement, or by using an integrating measuring instrument, only has makes sense if you already know from the outset that the quantity to be measured is exactly constant. But how can one know this if one has not first made a corresponding measurement in order to make this determination? If you really try to do this, you will get a dependency of the deflection on time (because of the Brownian fluctuations) with repeated observation or with the registration and therefore you cannot state with certainty whether the variable to be observed has remained exactly constant or is within the range limits of the registered variation has changed over time. This vicious circle is the reason why the method proposed to increase the measurement accuracy cannot actually be carried out.

In fact, we can even say with certainty that the requirement of constancy of J corresponding to the measurement process mentioned is certainly not fulfilled, since every macroscopically defined quantity that can be measured by a macroscopic measuring instrument is subject to fluctuations. For example, in reality there is certainly no exactly constant emf, even if the current source is protected with all possible refinement from external disturbances, because of the existence of spontaneous potential fluctuations due to the thermal motion of the electrons, which have recently been experimentally demonstrated by various researchers 1 Measuring an emf as precisely as possible therefore obviously means registering its dependency on time as precisely as possible or measuring the emf and its rate of change as precisely as possible over a short period of

time. This movement of the measuring instrument has an upper limit which is independent of the manner in which the measurement is carried out.

6. As has been repeatedly mentioned, the results reported in the preceding paragraphs can be traced back to the formal agreement between fusion theory and quantum mechanics, as is evident in particular in the comparison of equations (8) and (9) of §1 already drawn attention to the essential differences of a formal nature between the two equations. We will now try to understand the physical causes of these differences. At the same time, the following considerations are intended to contribute to the clarification of certain ambiguities that were recently highlighted by Ehrenfest,[1] with the call on physicists to take on these problems.

Classical diffusion can be considered as a flow governed, as we saw in §1, by a differential equation of the form (3), where F denotes a real differential operator and u a real function of place and time, which denote the density of the diffusing substance. From this it follows that it must be possible to calculate the density distribution at some later (and just as naturally also at an earlier) point in time from the specification of u at any point in time. In contrast to a problem of ordinary hydrodynamics, the diffusion flow in the system under consideration is completely determined by specifying the density as a function of the coordinates at any time, without the flow velocity, *i.e.*, a function of the coordinates, needing to be known at the same time. This is because the flow rate defined by equation (32) is a function of u and the coordinates alone and does not depend on the history of the system. If $u(x,y,z)$ is known, then $v(x,y,z)$ is also given by itself and therefore the change of the system in the following time element is completely determined in the sense of classical hydrodynamics.

We also note that a reversal of the time sequence, an interchange of t and -t in equation (3), is not possible since D is essentially positive. The process of diffusion is therefore “irreversible”. This is also evident from the fact mentioned that the velocity vector for a given u is a pure function of location, so the initial velocity is irreversible and is determined solely by the collisions of the surrounding molecules.

In the quantum mechanical case, the situation is completely different. Since the movement of the particles is not disturbed by collisions of the molecules of the surrounding substance, the movement of the particle swarm is essentially determined by the initial positions and the initial velocities of the particles. It is therefore clear that a differential equation for the density function w, as it applies to classical diffusion, cannot exist here. The simplest way to see that there is a

differential equation of type (4) for the complex function ψ is to look at the problem from the point of view of wave mechanics. In this view, the particle swarm is a "wave packet", *i.e.*, a superposition of harmonic partial waves of the shape

$$\psi_k = \phi_k e^{\frac{iE_k t}{\hbar}}$$

The number of which is generally infinitely large under the boundary conditions considered here. Here φ_k is the "amplitude function", a complex function of location of the form

$$\phi_k = a_k e^{iS_k}$$

Which contains the two real, scalar function of position that contains amplitude A_k and phase S_k.

By specifying all A_k and S_k as functions of position at a given point in time, the φ belonging to the wave packet under consideration is completely determined and therefore, according to the differential equation (4), also for any later (or earlier) point in time, which is physically obvious, since fate of each partial wave is completely determined by the indication of amplitude and phase at zero time and therefore also the fate of the wave packet created by interference from the partial waves. It is therefore immediately understandable that two scalar functions or one complex function, namely the Schrödinger function, are required to describe the state of the wave field.

Since the density of the considered swarm (now viewed from the corpuscular point of view) according to equation (5) is given by $|\psi|$ alone, the specifiable ψ as a function of the location makes a more extensive statement, *i.e.*, about the distribution of the positions of the particles at the point in time under consideration. According to what was said above, since the fate of the swarm is determined by ψ, a statement about the distribution of the velocities of the particles at the same time must obviously also be contained in the specification of ψ. However, if the initial velocities are not known, then one cannot make any predictions about the movement of the particle swarm from the initial distribution alone. So there can indeed not be a differential equation for $|\psi|$. Nevertheless, $w = |\psi|^2$ the density, or, interpreted in terms of a virtual totality, the probability density of the position alone is observable, but not ψ itself. This paradoxical fact can be explained immediately as a consequence of the uncertainty relations. If ψ itself were observable, then, according to what has been said, the position and velocity distribution in our particle swarm would be given at the same time, which is not possible!

That the coefficient on the left-hand side of equation (4) must be purely imaginary or the diffusion coefficient ε in (9) must be purely imaginary can be seen as follows: If at any time the phases S_k of all partial waves are reversed by 180º, then all φ_k become φ_k* and therefore ψ become

ψ^*. At the same time, however, the reversal of all phases is equivalent to the fact that the movement of the wave packet is reversed. The exchange of ψ with its conjugate complex value ψ^* means nothing other than a reversal of the time sequence, and the differential equation (4), which ψ satisfies, must therefore remain unchanged if ψ with ψ^* and t with –t interchanged. In fact, this is only possible, provided that the Hamiltonian H does not contain time, if the coefficient of $\partial\psi/\partial t$ is purely imaginary. The appearance of the imaginary diffusion coefficient means, as Schrödinger[1] has already pointed out, simply the reversibility of quantum mechanical "diffusion" as opposed to the classical one, a contradiction which we have already seen in §2 and 3 in the difference between equations (13) and (29) has revealed.

Prag, Januar 1933